

Infrared factorization in inclusive B meson decays

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Abstract

We perform infrared factorization of differential rates of radiative and semileptonic inclusive decays of the B meson in the end-point region of photon and charged lepton spectrum, respectively, in the leading heavy quark mass limit. We find that the differential rates are expressed in terms of hard, jet and soft functions, which satisfy evolution equations. Solving these equations, we find expressions for the moments of the differential rates in their end-point regions, which take into account all leading and nonleading logarithmic corrections in perturbation theory, as well as large nonperturbative power corrections. Expanded to the one-loop level, our predictions coincide with the results of existing lowest order calculations for $B \rightarrow \gamma X_s$ and $B \rightarrow l\bar{\nu} X_u$. Nonperturbative corrections appear in our formalism from the boundary value of the soft function in the evolution equation. The soft function is a universal process-independent function, which describes the distribution of the b quark in the B meson. Its behavior in the end-point region is governed by the nonperturbative asymptotics of a Wilson line expectation value. By considering the contributions of infrared renormalons, we find an ansatz for the Wilson line, which leads to a Gaussian model for the heavy quark distribution function.

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1. Introduction

Radiative and semileptonic decays of B mesons near maximum photon and charged lepton energy, respectively, are of special interest for the determination of electroweak parameters and the detection of new physics. They also present special theoretical challenges [1] because of large perturbative and nonperturbative corrections from QCD.

Recently, Bigi *et al.* [2] and Neubert [3] analyzed nonperturbative corrections to these processes in the Λ/M expansion, with M the heavy quark mass and Λ the usual QCD scale. They found that the leading term in this expansion is described by the same universal function, $f(x)$, for both processes. This function is nonperturbative in origin, and describes the B meson decay outside the phase space available for the decay of a free b quark. They also related the first few moments of $f(x)$ to hadronic matrix elements in heavy quark effective theory (HQET). As pointed out in [4, 2, 5], however, perturbative corrections are also large in the end-point region $x = E/E_{max} \sim 1$ and need to be resummed. It was observed that the leading (double-logarithmic) corrections, $(\alpha_s \ln^2(1-x))^k$, exponentiate [4]. Clearly, a complete picture of inclusive B decay in the end-point region requires a unified treatment of perturbative and nonperturbative contributions. Our goal in this paper is to develop such a formalism, based on well-known factorization theorems and evolution equations in perturbative QCD [6], and on the analysis of infrared renormalons [7]–[9] in the evolution of Wilson lines, or path-ordered exponentials.

In Section 2, we apply the factorization technique [10, 11, 12] to inclusive radiative decay $B \rightarrow \gamma X_s$ in the photon end-point region. The relevant factorization is valid to leading power in $1/M$, and involves two functions, one of which is the distribution $f(x)$ describing soft gluon emission by the b quark, and the other a “jet” function $J(x)$, describing collinear interactions of the outgoing s quark. The “universality” of $f(x)$ is a direct consequence of factorization. In Sections 3 and 4 we show the close relation of the distribution and jet functions to Wilson lines. We then derive the evolution of these functions, which resums both leading and nonleading logarithms as $x \rightarrow 1$. These equations are analogous to the usual GLAP [13] evolution equations, and follow directly from the renormalization properties of Wilson lines and quark fields. The solutions to these equations give an expression for $d\Gamma_{B \rightarrow \gamma X_s}/dx$ that includes all large perturbative corrections near the end-point. As in most evolution equations, they require a boundary value, which summarizes nonperturbative corrections. In Section 5, we show that the perturbative solution contains infrared renormalons, which introduce ambiguities in perturbation theory. These ambiguities occur precisely in the powers $\Lambda^2/[(1-x)M]^2$ that we expect from nonperturbative effects. This analysis lead us to a nonperturbative estimate for the relevant Wilson lines, and as a consequence, to a Gaussian model for the nonperturbative behavior of the distribution function. Finally, we extend our analysis to the end-point region in semileptonic decays, and show that the only significant difference is that these decays require an extra integral over the phase space of the neutrino.

2. Factorization in radiative decay of B-meson

Let us consider the inclusive radiative decay $B \rightarrow \gamma X_s$ in the limit when the mass of the s quark is neglected. In the rest frame of the B meson we define a scaling variable x as the ratio of the

photon energy to the mass of the b quark,

$$x = \frac{2E_\gamma}{M}.$$

The condition $M_X^2 > 0$ for the final state X_s leads to the restriction [2, 3],

$$0 < x < M_B/M \sim 1.09,$$

where M_B is the mass of the B meson. We are interested in the inclusive distribution, $\frac{d\Gamma}{dx}$, with the energy of photon near the end-point, $x \sim 1$. The decay $B \rightarrow \gamma X_s$ occurs through the transition $b \rightarrow s\gamma$, which is described by an effective hamiltonian [14]. In the heavy quark limit, $M \rightarrow \infty$, the momentum of the b quark is $Q_\mu = Mv_\mu$, where v_μ is the velocity of the B meson. As was stressed in [2, 3], perturbation theory describes a free b quark decay in the region $x < 1$, and in order to penetrate inside the “window” $1 < x < M_B/M$ one has to take into account nonperturbative effects. In the rest frame of the B meson, the momenta of the photon and b quark, q_μ and Q_μ , respectively, may be taken to have the following light-cone components ($q_\pm = \frac{1}{\sqrt{2}}(q_0 \pm q_3)$ and $\vec{q}_t = (q_1, q_2)$)

$$q_\mu \equiv (q_+, q_-, \vec{q}_t) = \frac{M}{\sqrt{2}}(x, 0, \vec{0}), \quad Q_\mu = \frac{M}{\sqrt{2}}(1, 1, \vec{0}). \quad (1)$$

Then, the momenta of the s quark, $P = Q - q$, is given by

$$P_\mu = \frac{M}{\sqrt{2}}(1 - x, 1, \vec{0}).$$

In the end-point region, $x \rightarrow 1$, the s quark moves in the minus direction with a high energy $\sim M/2$ and has a small invariant mass, $P^2 = M^2(1 - x) \ll M^2$. Thus, for $x \sim 1$, the s quark can produce a jet of collinear particles accompanied by soft radiation, from which we expect large perturbative and nonperturbative corrections. This is exactly the same situation one encounters analyzing the behaviour of hadronic processes (DIS structure functions, Drell-Yan cross section) in perturbative QCD near the boundary of phase space [10]. That is why the resummation technique developed in [10, 11, 12] applies to the inclusive B decay in the end-point region. In particular, analyzing diagrams of fig.1 contributing to the decay $B \rightarrow \gamma X_s$ we find the following three configurations of particle momenta associated with leading $1/M$ behavior (analog of leading twist in DIS):

$$\begin{aligned} \text{Hard}(H) : & \quad k_+ \sim k_- \sim k_t = \mathcal{O}(M), \\ \text{Jet}(J) : & \quad k_+ = \mathcal{O}(M(1 - x)), \quad k_- = \mathcal{O}(M), \quad k_t = \mathcal{O}(M\sqrt{1 - x}), \\ \text{Soft}(S) : & \quad k_+ \sim k_- \sim k_t = \mathcal{O}(M(1 - x)). \end{aligned}$$

For the $x \rightarrow 1$ limit, the hard subprocess gets contributions only from virtual particles. Notice that the minus and transverse components of the momenta of particles in the jet subprocess are much larger than those in the soft subprocess. That is, the jet subprocess carries almost all the P_- and P_t momentum, while the small P_+ momentum is distributed between the jet and soft subprocesses. In individual diagrams of fig.1, particles from jet and soft subprocess interact with each other. In the sum of all diagrams in the leading $1/M$ limit, however, the contribution of hard, jet and soft subprocesses may be factorized into the form [6, 10]

$$\frac{1}{\Gamma_\gamma} \frac{d\Gamma}{dx} = \frac{M}{v_+} \int_{(l_+)_{min}}^{P_+} dl_+ S(l_+) J(P_+ - l_+) H(P_-), \quad P_+ = \frac{M}{\sqrt{2}}(1 - x). \quad (2)$$

Here, $\Gamma_\gamma = \Gamma(b \rightarrow s\gamma) = \frac{\alpha_G^2}{32\pi^4} M^5 |V_{tb}V_{ts}^*|^2 C_7^2(M)$ is the partonic total width in the Born approximation [15], and l_+ is the light-cone component of the total momentum of soft gluons emitted in the partonic subprocess. Let us make an important comment about the integration limits for l_+ in (2). The upper limit $(l_+)_{max} = P_+$ follows from the condition that the momentum of the s quark jet, $P - l$, be time-like. The lower limit $(l_+)_{min}$ corresponds to the minimal energy of soft gluons in the final state. In perturbation theory soft gluons are emitted by b and s quarks into the final state and momentum conservation requires $(l_+)_{min} = 0$. In a realistic B meson, however, soft gluons may also take energy from the light components of the B meson. Thus, the minimal energy of soft gluons emitted by the b and s quarks in the partonic subprocess may even be negative nonperturbatively,

$$(l_+)_{min} = -(M_B - M)/\sqrt{2}, \quad (3)$$

which leads a window for $1 < x < M_B/M$.

To separate the subprocesses H , J and S in momentum space one has to introduce [6] a factorization scale μ . The contribution of each subprocess depends on this scale, although the μ -dependence cancels in the differential rate (2). In particular, the contribution of the hard subprocess, H , depends only on M and μ .¹

3. The soft subprocess

In emitting soft gluons, the b and s quarks behave as classical relativistic particles. That is, all effects of their interactions with soft gluons are factorized into eikonal phases given by the path-ordered exponentials [16, 17], or Wilson lines, $P \exp(i \int_C dz \cdot A(z))$, evaluated along their classical trajectories C with the gauge field $A(z)$ describing the soft radiation. As a consequence, the contribution of the soft subprocess to the differential rate is given by a Fourier transformed expectation value of a Wilson line² [11, 18]

$$S(l_+) = M v_+ \int_{-\infty}^{\infty} \frac{dy_-}{2\pi} e^{iy_- l_+} W_C((v \cdot y)\mu), \quad W_C \equiv \langle B | P \exp \left(i \int_C dz \cdot A(z) \right) | B \rangle, \quad (4)$$

where the integration over y_- fixes the total momentum of soft gluons in the final state to be equal to l_+ and $|B\rangle$ denotes the B meson state. The integration path C in the definition of W_C is shown in fig.2. It goes from $-\infty$ to point 0 along the quark velocity v , then along the light-cone minus direction to point y_- , and then from y_- to $-\infty$ along $-v$.

The following comments are in order. Notice that according to the definition, $S(l_+)$ depends only on the properties of the bound state $|B\rangle$ and not on the particular short distance subprocess. This is why $S(l_+)$ is a universal distribution. In fact, the function $S(l_+)$ was previously defined in [11] in the analysis of large x behavior of the structure function of DIS. It can be easily checked that the heavy quark distribution function [2, 3] coincides with the definition of $S(l_+)$. Indeed, in the leading $1/M$ limit, heavy quark fields are equivalent to path-ordered exponentials [19], and making this identification one gets $S(l_+) = F(x) = f(k_+)$ with $l_+ = -\bar{\Lambda}x/\sqrt{2} = -k_+/\sqrt{2}$ in the notation of refs.[2] and [3], respectively.

¹In what follows the dependence of hard, soft and jet subprocesses on the coupling constant $\alpha_s(\mu)$ is implied.

²Notice that the gauge fields are ordered along the integration path C and not according to time. However, on different parts of C path-ordering implies time or anti-time ordering [18].

In perturbation theory, the Wilson line W_C obeys the following relations [11, 18],

$$W_C = W(y_- v_+ \mu - i\varepsilon) = W^*(y_- v_+ \mu - i\varepsilon) = W(-y_- v_+ \mu + i\varepsilon),$$

which ensures the reality of $S(l_+)$. Here, μ is the factorization scale and the “ $-i\varepsilon$ ” prescription comes from the analogous property of the free gluon propagator and is thus of perturbative origin. Let us define a dimensionless variable z and rewrite (4) as

$$S(l_+) \equiv f(z, M/\mu) = \int_{-\infty}^{\infty} \frac{d\sigma}{2\pi} e^{i\sigma(1-z)} W(\mu\sigma/M - i\varepsilon), \quad l_+ = \frac{M}{\sqrt{2}}(1-z). \quad (5)$$

where $\sigma = y_- v_+ M$. Then, the “ $-i\varepsilon$ ” prescription immediately leads to the spectral property [11]

$$f(z, M/\mu) = 0, \quad \text{for } z > 1,$$

which implies that there is no “window”, $z > 1$, in perturbation theory. For this window to appear one has to take into account nonperturbative corrections to the Wilson line W_C in (4).

The heavy quark distribution function defined above, $f(z, M/\mu)$, gets large perturbative corrections $\sim \left(\frac{\log^n(1-z)}{1-z}\right)_+$ from the region $z \rightarrow 1$ corresponding to $l_+ \rightarrow 0$ in (2). Analyzing the moments of $f(z, M/\mu)$ in perturbation theory we find the following remarkable relation [18]

$$f_n(M/\mu) \equiv \int_0^{M_B/M} dz z^{n-1} f(z, M/\mu) = W(-in\mu/M) + \mathcal{O}(1/n), \quad (6)$$

which means that the large n behavior of the moments of the heavy quark distribution function is given by the Wilson line expectation value W_C evaluated along the path C with the formal identification $y_- v_+ = -in/M$. This suggests that in the large n limit one has to treat W_C as a nonlocal functional of the gauge field [11, 18], rather than to expand it into a divergent power series in y_- using the operator product expansion [2, 3].

We now use the renormalization properties [20, 21, 22] of the Wilson line W_C as a function of μ , which depend on the particular form of the path C . The integration path C of fig.2 has two cusps at points 0 and y and a light-like segment in between. As a consequence, the light-like Wilson line W_C obeys the renormalization group equation [11, 18]

$$\mathcal{D} W_C = -\{\Gamma_{cusp}(\alpha_s) [\log(\rho - i\varepsilon) + \log(-\rho + i\varepsilon)] + \Gamma(\alpha_s)\} W_C, \quad \rho = y_- v_+ \mu e^{\gamma_E}, \quad (7)$$

where $\mathcal{D} \equiv \mu \frac{\partial}{\partial \mu} + \beta(g) \frac{\partial}{\partial g}$, γ_E is the Euler constant and the anomalous dimensions $\Gamma_{cusp}(\alpha_s)$ and $\Gamma(\alpha_s)$ are known to two-loop order [21, 18]. Solving this equation, and using the relation (5), we find for the moments of the distribution function

$$f_n(M/\mu) = \exp \left(\int_{\mu}^{M n_0/n} \frac{dk_t}{k_t} \left[2\Gamma_{cusp}(\alpha_s(k_t)) \log \frac{k_t n}{M n_0} + \Gamma(\alpha_s(k_t)) \right] \right) f_n^{(0)}, \quad (8)$$

where $n_0 = e^{-\gamma_E}$ and $f_n^{(0)}$ has a nonperturbative origin, as the boundary value of the Wilson line in the solution of the RG equation (7). From this expression, we find that the moments obey the following evolution equation [11, 18],

$$\mathcal{D} f_n(M/\mu) = - \left(2\Gamma_{cusp}(\alpha_s) \log \frac{\mu n}{M n_0} + \Gamma(\alpha_s) \right) f_n(M/\mu), \quad (9)$$

where the r.h.s. depends on the normalization point through $\log \frac{n\mu}{n_0 M}$.

4. The jet subprocess

The jet subprocess J in eq.(2) describes the decay of the s quark with momentum $P - l$ into a jet of collinear particles. The invariant mass of the jet is $2P_-(P_+ - l_+) = M^2(z - x) \geq 0$, and the contribution of the jet subprocess depends only on this quantity and on the renormalization scale μ ,

$$J(P_+ - l_+) \equiv J(2P_-(P_+ - l_+), \mu^2) = J(M^2(z - x), \mu^2). \quad (10)$$

To take into account large perturbative corrections from the region $z \rightarrow x$ we again consider the moments:

$$J_n(M/\mu) \equiv M^2 \int_0^1 dz z^{n-1} J(M^2(1 - z), \mu^2). \quad (11)$$

After integration of the r.h.s., using the relation $z^n \approx e^{-n(1-z)}$ in the large n limit, the function J_n depends only on two scales, M^2/n and μ^2 , which allows us to write

$$J_n(M/\mu) = J_{n\mu^2/M^2}(1) + \mathcal{O}(1/n). \quad (12)$$

Let us now substitute expressions (5) and (10) into (2), using (3),

$$\frac{1}{\Gamma_\gamma} \frac{d\Gamma}{dx} = \int_x^{M_B/M} dz f(z, M/\mu) M^2 J(M^2(z - x), \mu^2) H(M/\mu), \quad (13)$$

and consider the moments of the differential rate defined as

$$\mathcal{M}_n(B \rightarrow \gamma X_s) \equiv \frac{1}{\Gamma_\gamma} \int_0^{M_B/M} dx x^{n-1} \frac{d\Gamma}{dx}. \quad (14)$$

The behavior of $d\Gamma/dx$ in the end-point region $x \sim 1$ corresponds to the large n asymptotics of the moments $\mathcal{M}_n(B \rightarrow \gamma X_s)$. We notice that for $z, x \sim 1$ one can replace $J(M^2(z - x), \mu^2) \approx J(M^2(1 - x/z), \mu^2)$ in (13). Then, the moments of the differential rate factorize [10] into the product of moments of the distribution function f_n and the moments of the jet J_n defined in (6) and (11),

$$\mathcal{M}_n(B \rightarrow \gamma X_s) = f_n(M/\mu) J_n(M/\mu) H(M/\mu) + \mathcal{O}(1/n) \quad (15)$$

The condition that the l.h.s. of this relation does not depend on the renormalization point μ leads to the following RG equations,

$$\begin{aligned} \mathcal{D} J_n(M/\mu) &= \left(2\Gamma_{cusp}(\alpha_s) \log \frac{\mu^2 n}{M^2 n_0} - 2\gamma(\alpha_s) \right) J_n(M/\mu), \\ \mathcal{D} H(M/\mu) &= \left(-2\Gamma_{cusp}(\alpha_s) \log \frac{\mu}{M} + 2\gamma(\alpha_s) + \Gamma(\alpha_s) \right) H(M/\mu), \end{aligned} \quad (16)$$

where the functional form of J_n , eq.(12), and independence of H on n were used. The anomalous dimensions entering into these equations have the following one-loop expressions [11, 18]

$$\Gamma_{cusp}(\alpha_s) = \frac{\alpha_s}{\pi} C_F, \quad \Gamma(\alpha_s) = -\frac{\alpha_s}{\pi} C_F, \quad \gamma(\alpha_s) = -\frac{3}{4} \frac{\alpha_s}{\pi} C_F. \quad (17)$$

Here, $\gamma(\alpha_s)$ coincides with the one-loop quark anomalous dimension in the axial gauge. To understand this property we notice that the collinear subprocess J can be defined as a cut propagator of the s quark in the light-like axial gauge $(n \cdot A) = 0$ with $n_\mu = (0, y_-, \vec{0})$. Indeed, in

this gauge the interaction between soft gluons and particles from the jet subprocess is suppressed in each diagram of fig.1 and the factorization (2) is manifest. Then, the evolution equation (16) describes the renormalization properties of a cut quark propagator in the light-like axial gauge.

Solving the RG equations (8) and (16) for S_n , J_n and H , we find after some algebra our complete expression for the moments of the decay distribution,

$$\begin{aligned} \mathcal{M}_n(B \rightarrow \gamma X_s) &= f_n^{(0)} C_\gamma(\alpha_s(M)) \\ &\times \exp \left(- \int_{n_0/n}^1 dy \left[2 \int_{My}^{M\sqrt{y}} \frac{dk_t}{k_t} \Gamma_{cusp}(\alpha_s(k_t)) + \Gamma(\alpha_s(My)) + \gamma(\alpha_s(M\sqrt{y})) \right] \right) \end{aligned} \quad (18)$$

where $C_\gamma = 1 + \mathcal{O}(\alpha_s(M))$ takes into account $\log^0(n)$ corrections to \mathcal{M}_n .

5. Infrared renormalons in Wilson lines

For large n , the integrals in (18) get dominant contribution from the region $y \sim 1/n$, and for $n \sim M/\Lambda$ we encounter the singularities of the coupling constant. This means that perturbation theory becomes ill-defined in the end-point region of the photon spectrum, and we have to take into account nonperturbative corrections to $d\Gamma_{B \rightarrow \gamma X_s}/dx$. As discussed in [7], both perturbative and nonperturbative contributions to physical quantities are ambiguous at the level of power corrections and only their sum is unique [7, 23]. This allows us to understand the structure of nonperturbative effects by analyzing the ambiguity of the perturbation series associated with the so-called infrared renormalons.

In the end-point region we expect large nonperturbative corrections to the Wilson line entering the definition (4) of the distribution function. Let us examine the contribution of IR renormalons to W_C by performing an “improved” calculation of W_C in perturbation theory. Using the definition (4) we find the one-loop contribution to W_C [18] and use the nonabelian exponentiation theorem for path ordered exponentials [24] to get

$$W_C = \exp \left(C_F \mu^{4-D} \int \frac{d^D k}{(2\pi)^{D-2}} \alpha_s \delta_+(k^2) \left(\frac{y}{(yk)} - \frac{v}{(vk)} \right)^2 (1 - e^{i(yk)}) (1 - e^{-i(yk)}) \right), \quad (19)$$

where $y = (0_+, y_-, \vec{0})$, $d^D k = dk_+ dk_- d^{D-2} \vec{k}$, and where μ is the scale parameter of the dimensional regularization with $D = 4 - 2\varepsilon$. The choice of the argument of the coupling constant α_s is determined by higher order corrections to W_C to be the transverse momentum of the gluons [17, 25]. Now let us substitute $\alpha_s = \alpha_s(\vec{k}^2) = \int_0^\infty d\sigma \exp(-\sigma \beta_0 \log \frac{\vec{k}^2}{\Lambda^2})$ into (19) and perform the integration over gluon momenta to get

$$W_C = \exp \left(C_F \frac{(4\pi\mu^2/\Lambda^2)^\varepsilon}{\Gamma(1-\varepsilon)} \int_{\varepsilon/\beta_0}^\infty d\sigma (\Lambda y_- v_+)^{2\sigma\beta_0} \Gamma(-2\sigma\beta_0) \cot(\pi\sigma\beta_0) (1 - \sigma\beta_0) \right). \quad (20)$$

Because $(\Lambda y_- v_+)^{2\sigma\beta_0} = \exp(-\sigma/\alpha_s(1/y_- v_+))$, the exponent has a form of an inverse Borel transformation [7]–[9]. Integration over small Borel parameter σ corresponds to large transverse momenta \vec{k}^2 and gives rise to ultraviolet poles in ε . After their subtraction in the \overline{MS} scheme, one verifies that W_C satisfies the RG equation (7). We now recognize that away from $\sigma = \varepsilon/\beta_0$ the integrand in (20) has infrared renormalon singularities generated by the Γ -function at points $\sigma^* = \frac{1}{2\beta_0}, \frac{1}{\beta_0}, \frac{3}{2\beta_0}, \dots$. Thus, to give a meaning to the perturbative expansion one has to fix a

prescription for integrating these singularities [9]. Different choices of the prescription lead to results which differ in power corrections of the form $(\Lambda y_-)^{2\sigma^*\beta_0}$. The leading power corrections correspond to the left-most singularity. Note that the singularity of the Γ -function at $\sigma^* = \frac{1}{2\beta_0}$ is compensated by the cotangent, so that the leading IR renormalon appears at $\sigma^* = \frac{1}{\beta_0}$. Thus, resummed perturbation theory generates (but fails to describe uniquely) $\mathcal{O}(\Lambda^2 y_-^2)$ power corrections to the Wilson line W_C . At the same time, this means that nonperturbative effects should also contribute to $\mathcal{O}(\Lambda^2 y_-^2)$ power corrections to make the Wilson line

$$W_C = W_{pert.}(y_-)W_{nonpert.}(y_-)$$

well defined. In summary, the exponentiation of perturbative corrections to $W_{pert.}(y_-)$ implies that IR renormalons appear in the exponent of $W_{pert.}(y_-)$. Hence, in order to compensate their contribution one can choose the “minimal” ansatz for nonperturbative part $W_{nonpert.}(y_-)$:

$$W_{nonpert.}(y_-) = \exp\left(-(y_-v_+)^2/l^2 + \mathcal{O}(y_-^3)\right), \quad (21)$$

where l is a nonperturbative correlation length. Notice that the expansion in the exponent does not contain a term linear in y_- .

Let us test the ansatz of eq.(21) by performing an expansion of the Wilson line W_C defined by (4) in powers of y_- . We denote a straight Wilson line as $\Phi_v(x) = P \exp(i \int_{-\infty}^0 ds v \cdot A(vs + x))$ and apply the identity $P \exp(i \int_0^1 ds y_- A_+(ys)) = \exp(-y_- D_+) \exp(y_- \partial_+)$ to get

$$W_C = \langle B | \Phi_v^\dagger(y_-) P e^{i \int_0^1 ds y_- A_+(ys)} \Phi_v(0) | B \rangle = 1 + \sum_{k=1}^{\infty} \frac{(iy_-)^k}{k!} \langle B | \Phi_v^\dagger(0) (iD_+)^k \Phi_v(0) | B \rangle$$

where $D_\mu = \partial_\mu - iA_\mu$ is the covariant derivative in the quark representation. According to its definition, the Wilson line satisfies the relation $(v \cdot D)\Phi_v(x) = 0$, which is similar to the equation of motion of the heavy quarks in HQET [26]. This allows us to apply HQET machinery for the evaluation of coefficients in the expansion of W_C :

$$W_C = 1 + a_1(iy_-v_+) + a_2(iy_-v_+)^2 + a_3(iy_-v_+)^3 + \dots, \quad (22)$$

where

$$a_1 = v_+^{-1} \langle B | \Phi_v^\dagger(iD_+) \Phi_v | B \rangle = 0, \quad a_2 = -\frac{1}{6} \langle B | \Phi_v^\dagger(iD)^2 \Phi_v | B \rangle, \quad a_3 = -\frac{1}{18} \langle B | \Phi_v^\dagger v_\mu F_{\mu\nu} D_\nu \Phi_v | B \rangle.$$

We recognize that our ansatz (21) is consistent at $\mathcal{O}(y_-^2)$ with the small y_- expansion of W_C . Note that in the leading $1/M$ limit, the heavy quark fields h_v are proportional to the Wilson line Φ_v , which is why the coefficients a_n are equal to analogous fundamental parameters in HQET [2, 3]. In particular, we can identify the correlation length l entering into (21) as

$$l^2 = -a_2 = -6/\lambda_1 = 6/\mu_\pi^2. \quad (23)$$

where the value of $\mu_\pi^2 = 0.52 \pm 0.12(\text{GeV})^2$ was estimated using QCD sum rules in [27] (which implies a particular choice for the perturbation series). Substituting the nonperturbative estimate (21) for the Wilson line W_C into (5), we find a distribution function $f^{(0)}(x)$, which can be used as a nonperturbative input for the evolution equations (9). If we use only the small y_- expansion (22) we get an expression for $f^{(0)}(x)$ as a series of derivatives of $\delta(1-x)$ -function, equivalent

to that proposed in [2, 3], which requires us to guess the shape of the function in the end-point region $x \sim 1$. On the other hand, integrating the nonperturbative ansatz (21), inspired by the IR renormalon analysis, we find the function ³

$$f^{(0)}(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(1-x)^2}{2\sigma^2}\right), \quad \sigma^2 = \frac{2}{l^2 M^2} = \frac{\mu_\pi^2}{3M^2}, \quad (24)$$

which describes a Gaussian distribution around $x = 1$ with the width σ . Notice however, that since we neglected $\mathcal{O}(y_-^3)$ corrections to the exponent of W_C in (21), this function does not vanish outside the physical region $x > M_B/M$ and (24) is valid only at the vicinity of $x = 1$. To get a better approximation for the distribution function one needs a more realistic estimate for the Wilson line expectation value. Once we know the distribution function $f^{(0)}(x)$, we evaluate its moments, $f_n^{(0)} = \int_0^{M_B/M} dx x^{n-1} f^{(0)}(x)$, and substitute them into (8) and (18).

6. Factorization in semileptonic decay of the B meson

The factorization for $B \rightarrow \gamma X_s$ was based on the kinematical requirement that, near the end point of the photon spectrum, the s quark must form a jet with large energy and small invariant mass. The same situation appears in semileptonic decay $B \rightarrow l\bar{\nu} X_u$ in the end-point region of the charged lepton spectrum. In this case, the decay occurs through an effective local vertex $\bar{u}\gamma_\mu(1-\gamma_5)b \cdot \bar{l}\gamma_\mu(1-\gamma_5)\nu$. Let l and ν be the momenta of the leptons, and Q and P the momenta of the b and u quarks, respectively. Standard notations for the scaling variables in the rest frame of the B meson are,

$$x = \frac{2E_l}{M}, \quad y = \frac{W^2}{M^2}, \quad y_0 = \frac{2W_0}{M}, \quad (25)$$

where $W = l + \nu = Q - q$ is the momentum transferred to the leptons. Phase space for these variables is

$$0 \leq x \leq x_m = M_B/M, \quad 0 \leq y \leq x x_m, \quad y/x + x \leq y_0 \leq y/x_m + x_m. \quad (26)$$

In analogy to the radiative decay, we examine the energy and invariant mass of the jet created by the u quark,

$$P^2 = M^2(1 - y_0 + y), \quad P_0 = \frac{M}{2}(2 - y_0). \quad (27)$$

From the definition (26) of phase space we find that

$$P^2 \leq M^2(1 - x)(1 - y/x), \quad P_0 \leq \frac{M}{2}(1 - y) + \frac{M}{2}(1 - x)(1 - y/x). \quad (28)$$

We conclude that near the end point of the lepton spectrum, $x \rightarrow 1$, with $y < 1$, we meet the same kinematical situation as in the $B \rightarrow \gamma X_s$ decay. Thus, once again, in the leading $1/M$ limit the dominant contribution to the differential rate of the decay comes from diagrams containing hard, jet and soft subprocesses. As in the previous case, the contributions of these subprocesses

³The same function has been proposed in [3].

are factorized from the partonic subprocess $\sigma_0 \sim (Q \cdot \nu)(l \cdot P) = \frac{M^4}{4}(x - y)(y_0 - x)$, and for the triple differential rate we get an expression similar to (2),

$$\frac{1}{\Gamma_l} \frac{d^3\Gamma}{dx dy dy_0} = (x - y)(y_0 - x) \frac{M}{v_+} \int_{(l_+)_\text{min}}^{P_+} dl_+ S(l_+) J(P_+ - l_+) H(P_-). \quad (29)$$

where $\Gamma_l = \frac{G_F^2}{16\pi^3} |V_{ub}|^2 M^5$, and $(l_+)_{\text{min}}$ was defined in (3). Note that the prefactor in (29) suppresses the “dangerous” region $y \rightarrow x \sim 1$. In this region, P_0 is forced to zero by (28) and factorization fails because the outgoing quark has vanishing energy. We note that relations (2) and (29) were found in a frame, where the outgoing quark has momentum $P_+ \ll P_-$. For the radiative decay, this frame is fixed by the particular choice (1) of the photon momentum q . For a semileptonic decay, we use the same choice for the lepton momentum, $l = \frac{M}{\sqrt{2}}(x, 0, \vec{0})$. Let us express P_+ and P_- in terms of the scaling variables (25). Using the relation $P_+ \ll P_-$ we make the following approximation: $P_- \approx P_- + P_+ = \sqrt{2}P_0 = \sqrt{2}(M - W_0)$ and get

$$P_- = \frac{M}{\sqrt{2}}(2 - y_0), \quad P_+ = \frac{P_-^2}{2P_-} = \frac{M}{\sqrt{2}}(1 - \xi), \quad \xi = \frac{1 - y}{2 - y_0}, \quad (30)$$

where the scaling variable ξ is an analog of x in the radiative decay, eq.(2). However, in contrast to the previous case, ξ depends on y and y_0 , and to calculate the differential rate $d\Gamma_{B \rightarrow l\bar{\nu} X_u}/dx$ one has to integrate (29) with respect to ξ in the physical region $x \leq \xi \leq M_B/M$. Using the kinematic relations (30) and the definitions (5) and (10) of the subprocesses we can rewrite (29) as

$$\frac{1}{\Gamma_l} \frac{d^3\Gamma}{dx dy dy_0} = (x - y)(y_0 - x) \int_{\xi}^{M_B/M} dz f\left(z, \frac{M}{\mu}\right) M^2 J(M^2(2 - y_0)(z - \xi), \mu^2) H\left(\frac{M(2 - y_0)}{\mu}\right). \quad (31)$$

Notice that if we replace subprocesses by their lowest order expressions $H = 1$, $f = f^{(0)}(z)$, $J = \delta(M^2(2 - y_0)(z - \xi))$, we get a result which contains all nonperturbative corrections, and which coincides with analogous expression in [2].

In order to analyze large perturbative corrections to $d\Gamma/dx$ in the end-point region, we change variables from y to ξ , and integrate (31) with respect to ξ and y_0 , neglecting terms that vanish as $x \rightarrow 1$,

$$\begin{aligned} \frac{1}{\Gamma_l} \frac{d\Gamma}{dx} &= \int_1^2 dy_0 (2 - y_0)^2 (y_0 - 1) \int_x^{M_B/M} d\xi \int_{\xi}^{M_B/M} dz \\ &\quad \times f\left(z, \frac{M}{\mu}\right) M^2 J(M^2(2 - y_0)(z - \xi), \mu^2) H\left(\frac{M(2 - y_0)}{\mu}\right) + \mathcal{O}(1 - x). \end{aligned} \quad (32)$$

Comparing this expression with (13) we notice that $-\frac{d}{dx}\left(\frac{1}{\Gamma_l} \frac{d\Gamma}{dx}\right)$ is similar, apart from y_0 -integral, to the differential rate $\frac{1}{\Gamma_\gamma} \frac{d\Gamma}{dx}$ of the radiative decay, eq.(13). This suggests the appropriate moments for semileptonic decay in analogy with (14),

$$\mathcal{M}_n(B \rightarrow l\bar{\nu} X_u) \equiv - \int_0^{M_B/M} dx x^n \frac{d}{dx} \left(\frac{1}{\Gamma_l} \frac{d\Gamma}{dx} \right) = \frac{n}{\Gamma_l} \int_0^{M_B/M} dx x^{n-1} \frac{d\Gamma}{dx}. \quad (33)$$

As in the case of radiative decays, we evaluate the moments of $\frac{d\Gamma}{dx}$, eq.(32), and find in the large n limit the following relation

$$\mathcal{M}_n(B \rightarrow l\bar{\nu} X_u) = f_n\left(\frac{M}{\mu}\right) \int_0^1 dx_\nu x_\nu (1 - x_\nu) J_n\left(\frac{M\sqrt{1 - x_\nu}}{\mu}\right) H\left(\frac{M(1 - x_\nu)}{\mu}\right) + \mathcal{O}(1/n) \quad (34)$$

where we have changed variables to $y_0 = 1 + x_\nu \approx 1 + \frac{2E_\nu}{M}$ with E_ν the energy of outgoing neutrino in the rest frame of the B meson. Here, the moments f_n and J_n are *identical* to those in the radiative decay and satisfy the evolution equations (9) and (16). The only difference with (15) is that one has to integrate in (34) with respect to the energy of the neutrino, x_ν , and take into account the x_ν -dependence of the collinear and hard subprocesses. This additional integration does not affect leading double logarithmic corrections $(\alpha_s \log^2 n)^k$, but it does affect nonleading terms. Note that in (34) the prefactor suppresses the end-points $x_\nu = 0$ and $x_\nu = 1$, which correspond to the limits of soft neutrino and outgoing quark, respectively. Thus, our factorized expression (34) takes care by itself near the “dangerous” point $x_\nu = 1$, where IR factorization fails.

Let us compare our predictions (15) and (34) for resummed large perturbative corrections with the results of one-loop calculations [28, 15] of the differential rate for $B \rightarrow \gamma X_s$ and $B \rightarrow l\bar{\nu} X_u$ in the end-point region. Solving the evolution equations (9) and (16) we find one-loop expressions for the subprocesses:

$$\begin{aligned} f_n(M/\mu) &= 1 + \frac{\alpha_s}{\pi} C_F \left(-\log^2 \frac{\mu n}{M n_0} + \log \frac{\mu n}{M n_0} \right) \\ J_n(M/\mu) &= 1 + \frac{\alpha_s}{\pi} C_F \left(2 \log^2 \frac{\mu \sqrt{n}}{M \sqrt{n_0}} + \frac{3}{2} \log \frac{\mu \sqrt{n}}{M \sqrt{n_0}} \right) \\ H(M/\mu) &= 1 + \frac{\alpha_s}{\pi} C_F \left(-\log^2 \frac{\mu}{M} - \frac{5}{2} \log \frac{\mu}{M} \right) \end{aligned} \quad (35)$$

where we have omitted constant terms and $f_n^{(0)}$. Substituting these relations into (15) and (34), we obtain the one-loop expression for the moments of the differential rates

$$\mathcal{M}_n \sim 1 + \frac{\alpha_s}{2\pi} C_F \left(-\log^2 n + A \log n + \text{const.} \right) \quad (36)$$

where coefficients in front of nonleading $\log n$ -term are

$$A_{B \rightarrow \gamma X_s} = \frac{7}{2}, \quad A_{B \rightarrow l\bar{\nu} X_u} = \frac{31}{6} \quad (37)$$

in accordance with the one-loop results of ref.[28] and [15].

Using the evolution equations (9) and (16), one can represent the expression (34) in a form similar to (18). It is more interesting however, to consider the ratio of the moments $\mathcal{M}_n(B \rightarrow \gamma X_s)$ and $\mathcal{M}_n(B \rightarrow l\bar{\nu} X_u)$ defined in (15) and (34), respectively. We find that the moments of the heavy quark distribution function, f_n , cancel in the ratio and we get

$$\begin{aligned} \frac{\mathcal{M}_n(B \rightarrow l\bar{\nu} X_u)}{\mathcal{M}_n(B \rightarrow \gamma X_s)} &= \frac{C_l}{C_\gamma} \int_0^1 dx_\nu x_\nu (1 - x_\nu) \exp \left(2 \int_{1-x_\nu}^1 \frac{dy}{y} \int_{M\sqrt{\frac{yn_0}{n}}}^{My} \frac{dk_t}{k_t} \Gamma_{cusp}(\alpha_s(k_t)) \right) \\ &\times \exp \left(\int_{1-x_\nu}^1 \frac{dy}{y} \left[-\gamma \left(\alpha_s \left(M \sqrt{yn_0/n} \right) \right) + 2\gamma(\alpha_s(My)) + \Gamma(\alpha_s(My)) \right] \right). \end{aligned} \quad (38)$$

where $C_l = 1 + \mathcal{O}(\alpha_s(M))$ is an analog of C_γ , defined in (18), for the case of semileptonic decay. Thus, all nonperturbative corrections cancel in the ratio of the moments of the differential rates and this allows us to calculate (38) perturbatively. We may then compare it with the ratio of experimental data using the definitions of the moments, (14) and (33), and isolating the ratios of the elements of the Cabibbo-Kobayashi-Maskawa matrix contained in the prefactors Γ_γ and Γ_l .

7. Conclusions

In this paper we performed infrared factorization on the differential rates of radiative and semileptonic inclusive decays of the B meson in the end-point regions of the photon and the charged lepton spectra. We found that in the leading $1/M$ limit the differential rates are expressed in terms of hard (H), jet (J) and soft (S) functions which satisfy evolution equations.

Solving the evolution equations we found expressions for the moments of the differential rates in the end-point region, (18) and (38), which take into account all leading and nonleading logarithmic $\log n$ corrections in perturbation theory, as well as large nonperturbative power corrections in the leading $1/M$ limit. Expanding these expressions in powers of the coupling constant and using one-loop results (17) for the anomalous dimensions entering the evolution equations, we have shown that our predictions coincide to the lowest order with the results of one-loop calculations for both processes.

Nonperturbative corrections appear in our formalism from the boundary value of the soft function $f^{(0)}(x)$ in the evolution equation (9). The soft function is the universal process-independent function which describes the distribution of the b quark in the B meson. We established that the behavior of $f^{(0)}(x)$ in the end-point region $x \sim 1$ is governed by the nonperturbative asymptotics of the Wilson line expectation values. Considering the contribution of infrared renormalons, we found a nonperturbative ansatz for the Wilson line which led to a Gaussian model (24) for the heavy quark distribution function $f^{(0)}$.

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References

- [1] A.Le Yaouanc, L.Oliver, O.Pene and J.C.Raynal, Talk given at 29th Rencontres de Moriond: QCD and High Energy Hadronic Interactions, Meribel les Allues, France, 19-26 Mar 1994; hep-ph/9406341.
- [2] I.I.Bigi, M.A.Shifman, N.G.Uraltsev and A.I.Vainshtein, preprint CERN-TH.7129/93, Dec. 1993; hep-ph/9312359; M.Shifman, preprint TPI-MINN-94-17-T, Feb 1994; hep-ph/9405246.
- [3] M.Neubert, Phys. Rev. D49 (1994) 4623, 3392.
- [4] G.Altarelli, N.Cabibbo, G.Corbo, L.Maiani and G.Martinelli, Nucl. Phys. B208 (1982) 365.
- [5] A.F.Falk, E.Jenkins, A.V.Manohar and M.B.Wise, Phys. Rev. D49 (1994) 4553.
- [6] J.C.Collins, D.E.Soper and G.Sterman, "Factorization of Hard Processes in QCD", in *Perturbative Quantum Chromodynamics*, ed. by A.H.Mueller (World Scientific, Singapore, 1989) p.1.

- [7] A.H.Mueller, Nucl. Phys. B250 (1985) 327; in *QCD 20 years later*, Aachen, 1992, Eds. P.M.Zerwas and H.A.Kastrup, World Scientific, Singapore, 1993.
M.Beneke and V.I.Zakharov, Phys. Lett. B312 (1993) 340;
V.I.Zakharov, Nucl. Phys. B385 (1992) 452.
- [8] M.Beneke and V.M.Braun, preprint MPI-PHT-94-9, Feb 1994; hep-ph/9402364;
A.V.Manohar and M.B.Wise, preprint UCSD-PTH-94-11, Jun 1994; hep-ph/9406392.
- [9] H.Contopanagos and G.Sterman, Nucl. Phys. B419 (1994) 77.
- [10] G.Sterman, Nucl. Phys. B281 (1987) 310;
S.Catani and L.Trentadue, Nucl. Phys. B327 (1989) 323; B353 (1991) 183.
G.P.Korchemsky and G.Marchesini, Phys. Lett. B313 (1993) 433
- [11] G.P.Korchemsky, Mod. Phys. Lett. A4 (1989) 1257.
- [12] S.V.Ivanov, G.P.Korchemsky and A.V.Radyushkin, Sov. J. Nucl. Phys. 44 (1986) 145;
G.P.Korchemsky and A.V.Radyushkin, Sov. J. Nucl. Phys. 45 (1987) 127, 910.
- [13] V.N.Gribov and L.N.Lipatov, Yad. Fiz. 15 (1972) 781; 1218;
L.N.Lipatov, Yad. Fiz. 20 (1974) 181;
G.Altarelli and G.Parisi, Nucl. Phys. 126B (1977) 298.
- [14] M.A.Shifman, A.I.Vainshtein and V.I.Zakharov, Phys. Rev. D18 (1978) 2583
- [15] G.Corbo, Nucl. Phys. B212 (1983) 99;
M.Jezabek and J.H.Kühn, Nucl. Phys. B320 (1989) 20.
- [16] M.Ciafaloni and G.Curci, Phys. Lett. B102 (1981) 352;
J.C.Collins and D.E.Soper, Nucl. Phys. B193 (1981) 381.
- [17] A.Bassetto, M.Ciafaloni and G.Marchesini, Phys. Rep. 100 (1983) 201.
- [18] G.P.Korchemsky and G.Marchesini, Nucl. Phys. B406 (1993) 225.
- [19] G.P.Korchemsky and A.V.Radyushkin, Phys. Lett. B279 (1992) 359.
- [20] A.M.Polyakov, Nucl. Phys. B164 (1980) 171;
I.Ya.Aref'eva, Phys. Lett. B93 (1980) 347;
V.S.Dotsenko and S.N.Vergeles, Nucl. Phys. B169 (1980) 527;
R.A.Brandt, F.Neri and M.-A.Sato, Phys. Rev. D24 (1981) 879.
- [21] G.P.Korchemsky and A.V.Radyushkin, Phys. Lett. B171 (1986) 459; Nucl. Phys. B283 (1987) 342.
- [22] I.A.Korchemskaya and G.P.Korchemsky, Phys. Lett. B287 (1992) 169;
A.Bassetto, I.A.Korchemskaya, G.P.Korchemsky and G.Nardelli, Nucl. Phys. B408 (1993) 62.
- [23] G.P.Korchemsky and G.Sterman, in preparation.

- [24] G.Sterman, in *Perturbative Quantum Chromodynamics*, (Tallahassee, 1981) Ed. D.W.Duke and J.F.Owens, AIP Conference Proceedings, No. 74, NY 1981, p.22.
J.G.M.Gatheral, Phys. Lett. 133B (1984) 90;
J.Frenkel and J.C.Taylor, Nucl. Phys. B246 (1984) 231.
- [25] D.Amati, A.Bassetto, M.Ciafaloni, G.Marchesini and G.Veneziano, Nucl. Phys. B173 (1980) 429.
- [26] H.Georgi, “Heavy Quark Effective Field Theory”, in *Boulder TASI Proceedings 91*, p.589 (QCD161:T45:1991).
- [27] P.Ball and V.M.Braun, Phys. Rev. D49 (1994) 2472.
- [28] A.Ali and C.Greub, Phys. Lett. B287 (1992) 191.

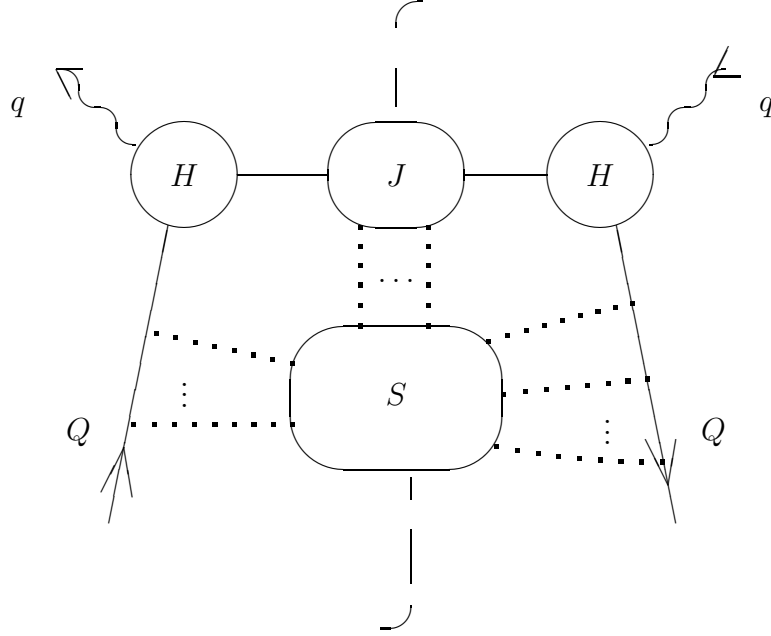


Fig.1: Unitarity diagram contributing to the differential rate of the radiative inclusive $B \rightarrow \gamma X_s$ decay in the leading $1/M$ limit in the axial gauge. We use solid lines for quarks, dotted lines for gluons, wave lines for photons and a dashed line for the final state.

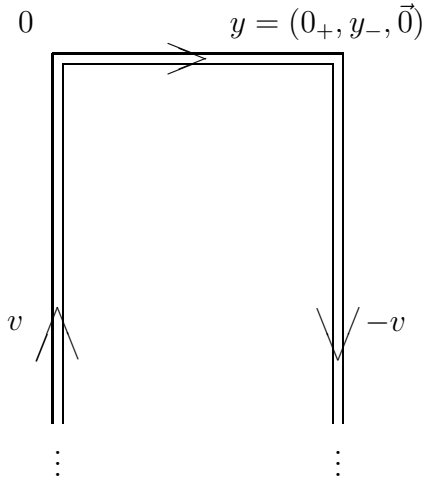


Fig.2: Integration path C entering into the definition (4) of the Wilson line W_C .